

CAYLEY FORM, COMASS, AND TRIALITY ISOMORPHISMS

MIKHAIL G. KATZ* AND STEVE SHNIDER

ABSTRACT. Following an idea of Dadok, Harvey and Morgan, we apply the triality property of $\text{Spin}(8)$ to calculate the comass of self-dual 4-forms on \mathbb{R}^8 . In particular, we prove that the Cayley form has comass 1 and that any self-dual 4-form realizing the maximal Wirtinger ratio (equation (1.5)) is $\text{SO}(8)$ -conjugate to the Cayley form. We also use triality to prove that the stabilizer in $\text{SO}(8)$ of the Cayley form is $\text{Spin}(7)$. The results have applications in systolic geometry, calibrated geometry, and $\text{Spin}(7)$ manifolds.

CONTENTS

1. Introduction	1
2. The Cayley form	5
3. Triality for D_4	7
4. Weight spaces in symmetric matrices and self-dual 4-forms	9
5. Proofs of Theorem 1.1 and Theorem 1.2	12
6. Stabilizer of the Cayley form	15
7. A counterexample	16
8. Acknowledgment	18
References	19

1. INTRODUCTION

The Cayley form, denoted ω_{Ca} , is a self-dual exterior 4-form on \mathbb{R}^8 . The form ω_{Ca} was first defined by R. Harvey and B. Lawson [HL82], by identifying \mathbb{R}^8 with the Cayley numbers (octonion algebra) and using well-known constructions of triple and quadruple vector cross products, see [BG67, Cu63, Kl63]. We observe that $\omega_{\text{Ca}} \in \Lambda^4 \mathbb{R}^8$ can

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be characterized in terms of an extremal property for the ratio of two norms, the comass norm and the Euclidean norm on $\Lambda^4 \mathbb{R}^8$. Namely, ω_{Ca} corresponds to a point of maximal Euclidean norm in the unit ball of the comass norm (see Section 2).

In systolic geometry [Gr83, Gr96, Gr99, Gr07, Ka07], the Cayley form plays a key role in the calculation of the optimal stable middle-dimensional systolic ratio of 8-manifolds, and in particular of the quaternionic projective plane, see [BKSW08]. For additional background on systolic geometry, see [Ka95, BK04, KL05, BCIK07, Bru08, DKR08].

The Cayley form defines an important case in the theory of calibrated geometries of Harvey and Lawson [HL82]. They remark that “the most fascinating and complex geometry discussed here is the geometry of Cayley 4-folds in $\mathbb{R}^8 \cong \mathbb{O}$ ”. The Cayley form is the calibrating form defining the Cayley 4-folds. In general, a k -form on a Riemannian manifold is called “calibrating” if it is closed and has pointwise comass 1.

The comass $\|\omega\|$ of a k -form ω on a normed vector space (such as the tangent space at a point on a Riemannian manifold) is defined as the maximum of the pairing with decomposable k -forms $v_1 \wedge \cdots \wedge v_k$ of norm 1:

$$\|\omega\| = \sup \left\{ \omega(v_1, \dots, v_k) \mid \forall i, |v_i| = 1 \right\}. \quad (1.1)$$

If ϕ is a calibrating k -form on \mathbb{R}^n with metric g , a k -dimensional subspace ξ is said to be calibrated by ϕ if $\phi|_{\xi} = \text{vol}_{(g|_{\xi})}$. A submanifold is said to be calibrated by a closed calibrating form ϕ if all of its tangent spaces are calibrated by ϕ . It follows immediately from the definition and Stokes theorem that a calibrated manifold minimizes volume within its homology class.

Research on calibrated geometries stimulated by [HL82] led to many new examples of spaces with exceptional holonomy. For example, the Cayley form is the basic building block in the structure of 8-manifolds with exceptional $\text{Spin}(7)$ holonomy, see [Jo00]. Major contributions in calibrated geometry and exceptional holonomy have been made by M. Berger, R. L. Bryant, D. Joyce, J. Dadok, F. R. Harvey, B. Lawson, F. Morgan and S. Salamon, [Ber55, Bry87, BryH89, BryS89, DHM88, M88, Sal89, Ha90, Jo96, Jo00, Jo07]. Riemannian manifolds with G_2 and $\text{Spin}(7)$ holonomy, of dimensions 7 and 8 respectively, are Ricci flat [Bo66]. The wealth of new examples of $\text{Spin}(7)$ and G_2 manifolds constructed by R.L. Bryant, D. Joyce, S. Salamon have been used as vacua for string theories, [Ac98, Be96, Le02, Sha95]. The Cayley 4-cycles on $\text{Spin}(7)$ manifolds are candidates for the supersymmetric representatives of fundamental particles [Be96].

A number of authors have calculated the comass $\|\omega_{\text{Ca}}\|$ of the Cayley form ω_{Ca} . Harvey and Lawson [HL82] used a definition of the Cayley form in terms of vector cross products of Cayley numbers. The basic identities they used are derived in a 7 page appendix. J. Dadok, R. Harvey, and F. Morgan [DHM88] studied the self-dual calibrations on \mathbb{R}^8 using triality, but their approach depends on a description of the geometry of polar representations [Da85].

In this paper, we give an explicit description (for certain weight spaces) of the intertwining operator between the triality related representations on traceless symmetric 8×8 matrices (see below) and on self-dual 4-forms on \mathbb{R}^8 . This allows us to use the representation of $\text{SO}(8)$ on traceless symmetric matrices to calculate the comass and describe the self-dual calibrations without appealing to the structure theorem for polar representations.

In addition to its relevance for calibrated geometry and special holonomy, the Cayley form is important for its applications in systolic geometry. To help understand the applications, we first recall the familiar case of 2-forms, which is to a certain (but limited) extent a model for what happens for 4-forms.

The space of alternating 2-forms on \mathbb{R}^n , identified with antisymmetric matrices on \mathbb{R}^n , becomes a Lie algebra with respect to the standard bracket $[A, B] = AB - BA$. An alternating 2-form α can be decomposed as a sum

$$\alpha = \sum_i c_i \alpha_i, \quad (1.2)$$

where the summands α_i are orthonormal, simple and commute pairwise, i.e. belong to a Cartan subalgebra, see Remark 1.4, item 2. Moreover, the summands can be chosen in such a way that the comass norm $\|\cdot\|$ as defined in (1.1), satisfies

$$\|\alpha\| = \max_i (|c_i|). \quad (1.3)$$

The standard Euclidean norm on \mathbb{R}^n extends to a Euclidean norm $|\cdot|$ on all the exterior powers, and we have

$$\frac{|\alpha|^2}{\|\alpha\|^2} \leq \text{rank}, \quad (1.4)$$

where “rank” is the dimension of the Cartan subalgebra. This optimal bound is attained by the standard symplectic form when $c_i = 1$ for all i .

It turns out that bounds similar to (1.4) remain valid for 4-forms on \mathbb{R}^8 , which are also part of a Lie algebra structure, defined below, but somewhat surprisingly, formula (1.3) is no longer true. See the

counterexample in Section 7. We will prove the following theorem, the first part of which was proved by different methods in [BKS08].

Theorem 1.1. *The Cayley form ω_{Ca} has comass 1 and satisfies the following relation:*

$$\frac{|\omega_{\text{Ca}}|^2}{\|\omega_{\text{Ca}}\|^2} = 14, \quad (1.5)$$

where the value 14 is the maximal possible value for 4-forms on \mathbb{R}^8 .

The approach using triality also leads to simple proofs of the following theorems.

Theorem 1.2. *Any self-dual 4-form on \mathbb{R}^8 satisfying (1.5) is $\text{SO}(8)$ -conjugate to the Cayley form.*

Theorem 1.3. *The subgroup of $\text{SO}(8)$ stabilizing the Cayley form is isomorphic to $\text{Spin}(7)$.*

Remark 1.4.

- (1) In Section 7 we give an example to show that a linear combination of the seven forms with all coefficients equal to +1, has comass 2, which shows that the situation for self-dual 4-forms on \mathbb{R}^8 is not completely parallel to the case of 2-forms, see equation (1.3).
- (2) In the course of the proof of Theorem 1.2, we prove that every self-dual 4-form on \mathbb{R}^8 is $\text{SO}(8)$ -conjugate to a linear combination of the following 7 mutually orthogonal self-dual forms:

$$\{e^{1234}, e^{1256}, e^{1278}, e^{1357}, e^{1467}, e^{1368}, e^{1458}\}, \quad (1.6)$$

where $e^{jklm} := e_j \wedge e_k \wedge e_l \wedge e_m + e_p \wedge e_q \wedge e_r \wedge e_s$ where the second summand is the Hodge dual of the first. The comment following (1.2) concerning a Cartan subalgebra is relevant here, because the 7 forms listed in (1.6) in fact form a maximal abelian subalgebra of real E_7 as defined in [Ad96, p. 76]. The conjugacy can be proved using this fact and a standard theorem in Lie theory.

- (3) Bryant [Bry87, p. 545] observed that $|\omega_{\text{Ca}}|^2 = 14$, but did not notice that this gave the maximal value for the norm of a calibrating 4-form.

One possible application is exploiting the \mathbb{R}^8 estimates described here so as to calculate the optimal stable middle-dimensional systolic ratio of 8-manifolds. Such an application depends on the existence of a Joyce manifold with middle-dimensional Betti number $b_4 = 1$. Currently, it is unknown whether such manifolds exist.

2. THE CAYLEY FORM

The Cayley form can be defined by two different coordinate-dependent constructions. There is also a coordinate-independent characterization of its $\mathrm{SO}(8)$ orbit.

Proposition 2.1. *We have the following three equivalent ways of describing the Cayley form ω_{Ca} :*

- (1) *The $\mathrm{SO}(8)$ orbit of ω_{Ca} consists of the set of points of the unit comass ball in $\Lambda^4(\mathbb{R}^8)$ of maximal Euclidean norm.*
- (2) *Under the identification of \mathbb{R}^8 with \mathbb{C}^4 , the Cayley form ω_{Ca} can be expressed as the sum of two terms, half the square of a standard Kahler form and the real part of a holomorphic volume form:*

$$\omega_{\mathrm{Ca}} = \frac{1}{2}\omega_J^2 + \mathrm{Re}(\Omega_J). \quad (2.1)$$

- (3) *Under the identification of \mathbb{R}^8 with $\mathbb{H} \oplus \mathbb{H}$, and quaternionic “vector space” structure given by right multiplication, the Cayley form is $\mathrm{SO}(8)$ -equivalent to the alternating sum of half the squares of the three Kähler forms associated with the complex structures given by right multiplication by i, j, k respectively, see [BryH89, Lemma 2.21]. If these forms are denoted ω_{J_a} , $a = 1, 2, 3$, then ω_{Ca} is $\mathrm{SO}(8)$ conjugate to*

$$\eta_2 = -\frac{1}{2}\omega_{J_1}^2 + \frac{1}{2}\omega_{J_3}^2 - \frac{1}{2}\omega_{J_2}^2. \quad (2.2)$$

Remark 2.2. The statement of item 1 was suggested to us by Blaine Lawson. The forms described in items 2 and 3 of the proposition correspond to two different points for the orbit described in item 1. Bryant and Harvey [BryH89] identify the Cayley form with the η_2 described in item 3. See Proposition 5.2 for the notation. The expression on the right side of equation (2.2) generalizes to n -dimensional quaternionic space for $n > 2$, and thus to hyper-Kähler manifolds. The Cayley form, denoted by Φ in [HL82, p. 120] and defined using octonions, is another point in the same orbit, η_3 in the notation of Proposition 5.2 below. The Cayley form is denoted ω_1 in [DHM88, p. 14], and Ω in [Jo00, p. 342].

Proof. The first assertion of the proposition is a consequence of Theorem 1.2. The proof is given in Section 5.

The simplest description of ω_{Ca} , the one given in item 2, is based on the standard identification of \mathbb{R}^8 with \mathbb{C}^4 .

Let $\{f_j\}$, $j = 1, \dots, 8$, be an orthonormal basis for \mathbb{R}^8 and $\{e_j\}$ the dual basis. Define a complex structure by

$$J(f_{2a-1}) = f_{2a}, \quad J(f_{2a}) = -f_{2a-1}, \quad a = 1, 2, 3, 4.$$

Then

$$\{e_{2a-1} + ie_{2a}, \quad a = 1, 2, 3, 4\} \quad (2.3)$$

form a basis for the complex linear dual space. The definition of the Cayley form, which uses standard constructions from complex differential geometry, is as follows. Define the symplectic form

$$\omega_J = \sum_{a=1, \dots, 4} e_{2a-1} \wedge e_{2a} = \frac{1}{2} \text{Im} \sum_{a=1, \dots, 4} (e_{2a-1} - ie_{2a}) \otimes (e_{2a-1} + ie_{2a}), \quad (2.4)$$

and the complex 4-form

$$\Omega_J = (e_1 + ie_2) \wedge (e_3 + ie_4) \wedge (e_5 + ie_6) \wedge (e_7 + ie_8); \quad (2.5)$$

then we define

$$\omega_{\text{Ca}} := \frac{1}{2} \omega_J^2 + \text{Re}(\Omega_J).$$

In terms of the dual basis $\{e_i | i = 1, \dots, 8\}$, the form ω_{Ca} is a signed sum of the 7 mutually orthogonal self-dual 4-forms

$$\{e^{1234}, e^{1256}, e^{1278}, e^{1357}, e^{1467}, e^{1368}, e^{1458}\},$$

where

$$e^{jklm} := e_j \wedge e_k \wedge e_l \wedge e_m + e_p \wedge e_q \wedge e_r \wedge e_s \quad (2.6)$$

and the second summand is the Hodge star of the first:

$$\omega_{\text{Ca}} := e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1368} - e^{1458} - e^{1467}, \quad (2.7)$$

see also (2.2).

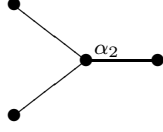
On $\mathbb{H} \oplus \mathbb{H}$, there are three Kähler forms defined by the three complex structures given by right multiplication by i, j, k respectively. They are

$$\begin{aligned} \omega_{J_1} &= e_1 \wedge e_2 - e_3 \wedge e_4 + e_5 \wedge e_6 - e_7 \wedge e_8, \\ \omega_{J_2} &= e_1 \wedge e_3 - e_4 \wedge e_2 + e_5 \wedge e_7 - e_8 \wedge e_6, \quad \text{and} \\ \omega_{J_3} &= e_1 \wedge e_4 - e_2 \wedge e_3 + e_5 \wedge e_8 - e_6 \wedge e_7. \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} \eta_2 &= e^{1234} - e^{1256} + e^{1278} - e^{1357} - e^{1368} - e^{1467} + e^{1458} \\ &= -\frac{1}{2} \omega_{J_1}^2 - \frac{1}{2} \omega_{J_2}^2 + \frac{1}{2} \omega_{J_3}^2. \end{aligned}$$

That η_2 is $SO(8)$ conjugate to ω_{Ca} follows from Proposition 5.2 and Theorem 1.2. \square


 FIGURE 3.1. Dynkin diagram of D_4 , see (3.1)

3. TRIALITY FOR D_4

The Lie group $\text{Spin}(8, \mathbb{R})$ has three 8-dimensional representations. They are the vector representation, $V = \mathbb{R}^8$, and the two spinor representations, Δ_+ and Δ_- . Fix a maximal torus $T \subset \text{Spin}(8)$, and a set of simple positive roots. Then for any automorphism $\phi \in \text{Aut}(\text{Spin}(8))$, the image $\phi(T)$ is another maximal torus. We can compose with a conjugation $\sigma_g(x) = gxg^{-1}$ so that $\sigma_g \circ \phi(T) = T$ and the fundamental chamber is preserved. In this way, an element of the outer automorphism group

$$\text{Out}(\text{Spin}(8)) = \text{Aut}(\text{Spin}(8))/\text{Inn}(\text{Spin}(8))$$

induces an automorphism of the Dynkin diagram D_4 of Figure 3.1. This correspondence determines an isomorphism with the symmetric group on three letters, $\text{Out}(\text{Spin}(8)) \cong \Sigma_3$, where the group Σ_3 permutes the three edges of the Dynkin diagram, see J.F. Adams [Ad96, pp. 33-36]. We identify $so(8)$ with 8×8 skew symmetric real matrices and the Cartan subalgebra $\mathfrak{h} \subset so(8)$ with the block diagonal matrices having four 2×2 blocks. An orthogonal basis $\{t_1, t_2, t_3, t_4\}$ is defined by the condition:

$$\sum x_i t_i = \text{diag}(x_1 J, x_2 J, x_3 J, x_4 J),$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while $\{x_i, i = 1, 2, 3, 4\}$ are coordinates in \mathfrak{h} .

The simple positive roots $\alpha_i \in \mathfrak{h}^*$ are

$$\alpha_1 = x_1 - x_2, \quad \alpha_2 = x_2 - x_3, \quad \alpha_3 = x_3 - x_4, \quad \alpha_4 = x_3 + x_4, \quad (3.1)$$

where α_2 appears at the center of the diagram of Figure 3.1. The fundamental weights $\lambda_i \in \mathfrak{h}^*$ are

$$\begin{aligned} \lambda_1 &= x_1, \\ \lambda_2 &= x_1 + x_2, \\ \lambda_3 &= \frac{1}{2}(x_1 + x_2 + x_3 - x_4), \\ \lambda_4 &= \frac{1}{2}(x_1 + x_2 + x_3 + x_4), \end{aligned}$$

and the corresponding representations are

$$\rho_1 \text{ on } \Lambda^1(V) = V, \quad \rho_2 \text{ on } \Lambda^2(V), \quad \rho_3 \text{ on } \Delta_-, \quad \rho_4 \text{ on } \Delta_+,$$

respectively. Let $\sigma^2(V)$ be the representation of $\text{Spin}(8)$ on the second symmetric power of V , which, by the $SO(8)$ equivalence of V and V^* , is equivalent to the representation by conjugation on the 8×8 traceless symmetric matrices. Let $\sigma_0^2(V)$ be the subrepresentation on the traceless symmetric matrices, so that one has a decomposition

$$\sigma^2(V) \cong \mathbf{1} \oplus \sigma_0^2(V).$$

The second symmetric power of Δ_+ decomposes as

$$\sigma^2(\Delta_+) = \mathbf{1} \oplus \Lambda_+^4(V),$$

where $\Lambda_+^4(V)$ is the representation of $\text{Spin}(8)$ on the self-dual 4-forms, see [Ad96, p. 25, Theorem 4.6].

The representations

$$\pi_2 : \text{Spin}(8) \rightarrow \text{Aut}(\sigma_0^2(V))$$

to

$$\pi_4 : \text{Spin}(8) \rightarrow \text{Aut}(\Lambda_+^4(V))$$

both factor through the vector representation,

$$\rho_1 : \text{Spin}(8) \rightarrow \text{SO}(8).$$

If $\hat{\pi}_2$ and $\hat{\pi}_4$ denote the respective $\text{SO}(8)$ representations

$$\hat{\pi}_2 : \text{SO}(8) \rightarrow \text{Aut}(\sigma_0^2(V)) \quad \text{and}$$

$$\hat{\pi}_4 : \text{SO}(8) \rightarrow \text{Aut}(\Lambda_+^4(V),$$

then

$$\pi_2 = \hat{\pi}_2 \circ \rho_1 \quad \pi_4 = \hat{\pi}_4 \circ \rho_1. \tag{3.2}$$

Let ϕ be the automorphism (preserving the maximal torus and fundamental chamber) representing the outer automorphism that interchanges the fundamental weights λ_1 and λ_4 , and leaves λ_3 fixed. Then ϕ transforms the representation π_2 to π_4 . In other words, there is a linear isomorphism $\psi : \sigma_0^2(V) \rightarrow \Lambda_+^4(V)$ such that

$$\psi(\pi_2(g)w) = \pi_4(\phi(g))\psi(w), \tag{3.3}$$

for all $g \in \text{Spin}(8)$ and $w \in \sigma_0^2(V)$. See Figure 3.2.

$$\begin{array}{ccc}
\sigma_0^2(V) & \xrightarrow{\pi_2(g)} & \sigma_0^2(V) \\
\downarrow \psi & & \downarrow \psi \\
\Lambda_+^4(V) & \xrightarrow{\pi_4 \circ \phi(g)} & \Lambda_+^4(V)
\end{array}$$

FIGURE 3.2. Intertwining of a pair of representations

4. WEIGHT SPACES IN SYMMETRIC MATRICES AND SELF-DUAL 4-FORMS

In this section we describe the map ψ in terms of the weights of $\sigma_0^2(V)$ and $\Lambda_+^4(V)$. Since we are dealing with real representations of a compact group, the weight spaces will be real two dimensional subspaces.

In the complexified representation $\sigma_0^2(V)$, the vector

$(e_{2a-1} + ie_{2a}) \otimes (e_{2a-1} + ie_{2a})$ is a weight vector with weight $2ix_a$.

In the real representation, we call the 2-dimensional real subspace with basis

$$u_a = e_{2a-1} \otimes e_{2a-1} - e_{2a} \otimes e_{2a}, \quad \text{and} \quad v_a = e_{2a-1} \otimes e_{2a} + e_{2a} \otimes e_{2a-1}$$

a weight space for the weight, $2x_a$, $a = 1, 2, 3, 4$.

In terms of traceless symmetric 8×8 matrices $so(8)$ acting by matrix commutator, the elementary formulae:

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

imply

$$[x_1 t_1 + x_2 t_2 + x_3 t_3 + x_4 t_4, u_a] = -2x_a v_a$$

and

$$[x_1 t_1 + x_2 t_2 + x_3 t_3 + x_4 t_4, v_a] = 2x_a u_a,$$

where

$$u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$u_a = \begin{pmatrix} 0_{2a-2} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0_{8-2a} \end{pmatrix}, \quad v_a = \begin{pmatrix} 0_{2a-2} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 0_{8-2a} \end{pmatrix}. \quad (4.1)$$

The proof of the following lemma is a straightforward calculation. Recall the notation from (2.6).

Lemma 4.1. *The real representation Λ_+^4 of $\mathrm{SO}(8, \mathbb{R})$ has highest weight $2\lambda_4 = (x_1 + x_2 + x_3 + x_4)$ corresponding to the matrix $(x_1 + x_2 + x_3 + x_4)J$ acting on the two dimensional real weight space with basis $\{\mu_1, \nu_1\}$, where*

$$\begin{aligned}\mu_1 &= \mathrm{Re}\Omega_J = e^{1357} - e^{1467} - e^{1368} - e^{1458}, \\ \nu_1 &= \mathrm{Im}\Omega_J = -e^{1468} + e^{1358} + e^{1457} + e^{1367},\end{aligned}\tag{4.2}$$

where Ω_J is defined in (2.5) Conjugating two of the complex linear factors $e_{2b-1} + ie_{2b}$ and $e_{2c-1} + ie_{2c}$ in Ω_J , (2.5), gives rise to weight spaces with weights having a coefficient -1 for x_b and x_c and coefficient $+1$ for the remaining x_a . Thus we define three other real weight spaces with bases $\{\mu_j, \nu_j\}$ and weights as listed below:

$$\mu_2 = e^{1357} + e^{1467} - e^{1368} + e^{1458} \text{ and } \nu_2 = -e^{1468} - e^{1358} + e^{1457} - e^{1367}\tag{4.3}$$

with weight $2(\lambda_2 - \lambda_4) = x_1 + x_2 - x_3 - x_4$;

$$\mu_3 = e^{1357} + e^{1467} + e^{1368} - e^{1458} \text{ and } \nu_3 = -e^{1468} - e^{1358} - e^{1457} + e^{1367}\tag{4.4}$$

with weight $2(\lambda_1 - \lambda_2 + \lambda_3) = x_1 - x_2 + x_3 - x_4$; and

$$\mu_4 = e^{1357} - e^{1467} + e^{1368} + e^{1458} \text{ and } \nu_4 = -e^{1468} + e^{1358} - e^{1457} - e^{1367}\tag{4.5}$$

with weight $2(\lambda_1 - \lambda_3) = x_1 - x_2 - x_3 + x_4$.

The decomposition in equation (2.1) expresses ω_{Ca} as a sum of a zero weight vector and a highest weight vector for π_4 .

The intertwining diagram in Figure 3.2 implies that ψ maps a weight space of the representation π_2 into the corresponding weight space for the representation $\pi_4 \circ \phi$. Since ϕ interchanges λ_1 and λ_4 :

- (1) the weight space for $2\lambda_1 = 2x_1$ in the representation $\pi_4 \circ \phi$ is the weight space for $2\lambda_4 = x_1 + x_2 + x_3 + x_4$ in the representation π_4 ,
- (2) the weight space for $2(\lambda_2 - \lambda_1) = 2x_2$ in the representation $\pi_4 \circ \phi$ is the weight space for $2(\lambda_2 - \lambda_4) = x_1 + x_2 - x_3 - x_4$ in the representation π_4 ,
- (3) the weight space for $2(\lambda_4 - \lambda_2 + \lambda_3) = 2x_3$ in the representation $\pi_4 \circ \phi$ is the weight space for $2(\lambda_1 - \lambda_2 + \lambda_3) = x_1 - x_2 + x_3 - x_4$ in the representation π_4

- (4) the weight space for $2(\lambda_4 - \lambda_3) = 2x_4$ in the representation $\pi_4 \circ \phi$ is the weight space for $2(\lambda_1 - \lambda_3) = x_1 - x_2 - x_3 + x_4$ in the representation π_4 .

If we conjugate ϕ by an element k , and multiply ψ by $\pi_4(k)$ equation (3.3) becomes

$$\pi_4(k\phi(g)k^{-1})\pi_4(k)\psi(v) = \pi_4(k)\psi(\pi_2(g)v). \quad (4.6)$$

Conjugating by an appropriate element of the maximal torus, we can rotate the basis in each weight space and assume

$$\psi(u_j) = \frac{1}{2}\mu_j, \quad (4.7)$$

for $j = 1, \dots, 4$, and u_j is defined by (4.1). The factor $\frac{1}{2}$ is required in order that ψ be an isometry.

Note that $\pi_4(k)\psi(z) = \psi(z)$ for k in the maximal torus and z a zero weight vector in $\sigma_0^2(V)$.

The zero weight space of $\sigma_0^2(V)$, when presented as matrices, is the three dimensional space with an orthogonal basis consisting of the matrices

$$z_1 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}, \quad z_2 = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & -I_4 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$$

$$z_3 = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & -I_2 \end{pmatrix}.$$

Lemma 4.2. *The intertwiner ψ acts on the 0 weight space as follows:*

$$\psi(z_1) = 2e^{1234}, \quad \psi(z_2) = 2e^{1278}, \quad \psi(z_3) = 2e^{1256}. \quad (4.8)$$

Proof. The involution ϕ leaves the simple root $\alpha_3 = x_3 - x_4$ invariant, and hence also the real 2 dimensional subspace which is a real form of the complex subspace of root vectors $E_{\pm\alpha_3}$, with a basis:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & J & 0 \end{pmatrix}. \quad (4.9)$$

The element

$$g_1 = \exp((\pi/2)E_1) \in \text{SO}(8)$$

acting in $\sigma_0^2(V)$ fixes z_1 and interchanges z_2 and z_3 , and acting in Λ_+^4 it fixes e^{1234} and interchanges e^{1256} and e^{1278} . In fact, the element g_1 acts in the coadjoint representation as reflection in α_3 . Since $\phi(g_1) = g_1$, the

image of z_1 under ψ must be a multiple of e^{1234} . The isometry condition implies $\psi(z_1) = \pm 2e^{1234}$. We normalize the multiple to $+2$, using $-\psi$ if necessary and another rotation, see equation (4.6), by an element of the maximal torus to guarantee that $\psi(u_i) = \frac{1}{2}\mu_i$, (4.7). The element $g_2 \in SO(8)$ acting in the coadjoint representation as Weyl reflection in α_2 is also invariant under ϕ . It interchanges z_1 and z_3 in the space of traceless symmetric matrices and interchanges e^{1234} and e^{1256} in the self-dual forms, so

$$\begin{aligned}\psi(z_3) &= \psi(\pi_2(g_2)z_1) \\ &= \pi_4(\phi(g_2))\psi(z_1) \text{ eq. (3.3)} \\ &= \pi_4(g_2)2e^{1234} \\ &= e^{1256}\end{aligned}$$

A similar argument using the element whose coadjoint action is reflection in $\alpha_2 + \alpha_3$ shows that $\psi(z_2) = 2e^{1278}$ and completes the proof of equation(4.8). \square

Putting together equations (4.7) and (4.8) define

$$A_1 := \begin{pmatrix} \frac{7}{8} & 0 \\ 0 & -\frac{1}{8}I_7 \end{pmatrix}, \quad (4.10)$$

and

$$\omega_{\text{Ca}} := e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1467} - e^{1368} - e^{1458}. \quad (4.11)$$

Then

$$\begin{aligned}\psi(A_1) &= \psi\left(\frac{1}{8}(z_1 + z_2 + z_3) + \frac{1}{2}(u_1)\right) \\ &= \frac{1}{4}(e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1467} - e^{1368} - e^{1458}) \\ &= \frac{1}{4}\omega_{\text{Ca}}.\end{aligned} \quad (4.12)$$

5. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

Proposition 5.1. *The self dual form*

$$\omega_{\text{Ca}} = e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1467} - e^{1368} - e^{1458}$$

has comass 1.

Proof. We need to prove that

$$\sup_{g \in SO(8)} (\omega_{\text{Ca}}, g(e_1 \wedge e_2 \wedge e_3 \wedge e_4)) = 1 \quad (5.1)$$

First of all, ω_{Ca} is self-dual and therefore orthogonal to the anti-self dual part of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$, so we have

$$(\omega_{\text{Ca}}, g(e_1 \wedge e_2 \wedge e_3 \wedge e_4)) = \frac{1}{2}(\omega_{\text{Ca}}, ge^{1234}).$$

Next,

$$\begin{aligned} \frac{1}{2}(\omega_{\text{Ca}}, ge^{1234}) &= \frac{1}{4}(\omega_{\text{Ca}}, g\psi(z_1)) \quad \text{by (4.8)} \\ &= \left(\frac{1}{4}\omega_{\text{Ca}}, g\psi(z_1)\right) \\ &= (\psi(A_1), g\psi(z_1)) \quad \text{by (4.12)} \\ &= (\psi(A_1), \psi(\phi(g)z_1)) \quad \text{by (3.3)} \\ &= (A_1, \phi(g)z_1), \end{aligned}$$

since ψ is an isometry. Now

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0_7 \end{pmatrix} - \frac{1}{8}I_8,$$

and $(I_8, \phi(g)z_1) = \text{trace}(\phi(g)z_1) = 0$. Putting this all together we have

$$\begin{aligned} \sup_{g \in \text{SO}(8)} (\omega_{\text{Ca}}, g(e_1 \wedge e_2 \wedge e_3 \wedge e_4)) &= \\ &= \sup_{g \in \text{SO}(8)} \left(A_1 + \frac{1}{8}I_8, \phi(g)z_1 \right) \\ &= \sup_{g' = \phi(g)^{-1} \in \text{SO}(8)} \left(g' \begin{pmatrix} 1 & 0 \\ 0 & 0_7 \end{pmatrix} g'^{-1}, z_1 \right) \\ &= \sup_{g' \in \text{SO}(8)} \left\{ \sum_{i=1, \dots, 4} (g'_{i1})^2 - (g'_{i+4,1})^2 \right\} \\ &= 1, \end{aligned}$$

proving the result. □

Proposition 5.2. *The following self dual forms all have comass 1:*

$$\begin{aligned}
\omega_2 &= 4\psi\left(\frac{1}{8}(z_1 + z_2 + z_3) - \frac{1}{2}u_1\right) \\
&= e^{1234} + e^{1256} + e^{1278} - e^{1357} + e^{1467} + e^{1368} + e^{1458} \\
\omega_3 &= 4\psi\left(\frac{1}{8}(z_1 - z_2 - z_3) + \frac{1}{2}u_2\right) \\
&= e^{1234} - e^{1256} - e^{1278} + e^{1357} + e^{1467} - e^{1368} + e^{1458} \\
\omega_4 &= 4\psi\left(\frac{1}{8}(z_1 - z_2 - z_3) - \frac{1}{2}u_2\right) \\
&= e^{1234} - e^{1256} - e^{1278} - e^{1357} - e^{1467} + e^{1368} - e^{1458} \\
\eta_1 &= 4\psi\left(\frac{1}{8}(z_1 + z_2 - z_3) - \frac{1}{2}u_3\right) \\
&= e^{1234} - e^{1256} + e^{1278} + e^{1357} + e^{1467} + e^{1368} - e^{1458} \\
\eta_2 &= 4\psi\left(\frac{1}{8}(z_1 + z_2 - z_3) + \frac{1}{2}u_3\right) \\
&= e^{1234} - e^{1256} + e^{1278} - e^{1357} - e^{1467} - e^{1368} + e^{1458} \\
\eta_3 &= 4\psi\left(\frac{1}{8}(z_1 - z_2 + z_3) - \frac{1}{2}u_4\right) \\
&= e^{1234} + e^{1256} - e^{1278} + e^{1357} - e^{1467} + e^{1368} + e^{1458} \\
\eta_4 &= 4\psi\left(\frac{1}{8}(z_1 - z_2 + z_3) - \frac{1}{2}u_4\right) \\
&= e^{1234} + e^{1256} - e^{1278} - e^{1357} + e^{1467} - e^{1368} - e^{1458}.
\end{aligned}$$

Proof. Let D_i be the diagonal matrix with 1 the i th position, all other entries 0, and $A_i := D_i - \frac{1}{8}I$. The expressions in parentheses on the right side of the equations above equal A_i for $i = 2, 3, 4$ and $-A_i$ for $i = 5, 6, 7, 8$. These matrices are $SO(8)$ conjugate, hence so are the corresponding self-dual 4-forms. \square

For all the forms $\nu = \omega_j$, or $\nu = \eta_j$, $j = 1, 2, 3, 4$ we have

$$\max_{g \in SO(8)} (\nu, g(e_1 \wedge e_2 \wedge e_3 \wedge e_4)) = (\nu, e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1. \quad (5.2)$$

Any convex combination of the ω_j, η_j also satisfies (5.2) and therefore has comass 1. Conversely, any self-dual 4-form satisfying (5.2) corresponds under triality to a traceless symmetric 8×8 matrix A satisfying

$$\max_{g \in SO(8)} (A, gz_1) = 1, \quad \text{and} \quad (A, z_1) = 1.$$

An elementary argument cf. [DHM88, Lemma 3.4], shows that any such matrix is a convex combination of the matrices

$$\{A_1, \dots, A_4, -A_5, \dots, -A_8\}.$$

Taking the image under ψ , we see that any self-dual 4-form ν , satisfying (5.2) is a convex combination of the ω_j, η_j . Any comass 1 self-dual 4-form is $\mathrm{SO}(8)$ -conjugate to one satisfying (5.2), which we have just shown to be a convex combination of the ω_j, η_j .

We will now prove Theorem 1.2, to the effect that every self-dual 4-form on \mathbb{R}^8 satisfying (1.5) is $\mathrm{SO}(8)$ -conjugate to the Cayley form.

Proof of Theorem 1.2. Let ω be a self-dual 4-form satisfying (1.5). We can assume that ω is normalized to unit comass. As noted above, the comass 1 condition implies that ω is conjugate to a convex combination

$$\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + a_5\eta_1 + a_6\eta_2 + a_7\eta_3 + a_8\eta_4, \quad (5.3)$$

with $a_i \geq 0$ and $\sum a_i = 1$. Since the $\{\frac{1}{\sqrt{14}}\omega_i, \frac{1}{\sqrt{14}}\eta_i\}$ form an orthonormal set, (5.3) implies

$$\frac{|\omega|^2}{14} = \sum a_i^2 \leq \sum a_i = 1,$$

with equality if and only if all the a_i except one are zero. Thus to achieve the maximum Euclidean norm 14, and satisfy (5.2), ω must be one of the 8 forms $\{\omega_j, \eta_j | j = 1, \dots, 4\}$ all of which are $\mathrm{SO}(8)$ -conjugate to the Cayley form. \square

6. STABILIZER OF THE CAYLEY FORM

In this section we give a proof using triality of Theorem 1.3 stating that the stabilizer of the Cayley form is $\mathrm{Spin}(7)$.

Proof. Recall, (3.3), that there is a linear isometry $\psi : \sigma_0^2(V) \rightarrow \Lambda_+^4(V)$ such that for all $g \in \mathrm{Spin}(8)$

$$\pi_4(\phi(g))\psi(v) = \psi(\pi_2(g)v), \quad \text{and} \quad \psi(A_1) = \omega_{\mathrm{Ca}},$$

where ϕ be the triality automorphism interchanging the fundamental weights λ_1 and λ_4 and A_1 is the diagonal matrix defined in the proof of Proposition 5.2. If G denotes the stabilizer of A_1 in the representation π_2 , then the stabilizer of ω_{Ca} in the representation π_4 is $\phi(G)$. Both representations π_2 and π_4 factor through $\rho_1 : \mathrm{Spin}(8) \rightarrow \mathrm{SO}(8)$. Composing with ρ_1 , we see that the stabilizer of ω_{Ca} in the representation $\hat{\pi}_4$ of $\mathrm{SO}(8)$ (see (3.2)) is $\rho_1\phi(G)$.

A simple matrix calculation shows that the $\mathrm{SO}(8)$ -stabilizer of $A_1 \in \sigma_0^2(V)$ is the subgroup $\mathrm{O}(7) \cong \{\pm I_8\} \times \mathrm{SO}(7) \cong \mathbb{Z}_2 \times \mathrm{SO}(7)$. Let γ be the “volume form” in the Clifford algebra:

$$\gamma = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8,$$

which is also an element of $\text{Spin}(8)$. In the vector representation $\rho_1(\gamma) = -I_8$; therefore,

$$G = \rho_1^{-1}(\{\pm I_8\} \times \text{SO}(7)) = \{1, \gamma\} \times \rho_1^{-1}(\text{SO}(7)) = \{1, \gamma\} \times \text{Spin}(7).$$

To complete the proof, we will show that $\rho_1\phi$ is injective on the subgroup $\text{Spin}(7)$, that is, $\text{Ker}(\rho_1\phi) \cap \text{Spin}(7) = \{1\}$.

The ± 1 eigenspaces of γ define the splitting of the Clifford module: $\Delta = \Delta_+ \oplus \Delta_-$, and the kernel of the representation $\rho_4 : \text{Spin}(8) \rightarrow \text{Aut}(\Delta_+)$ is $\{1, \gamma\}$.

Since ϕ conjugates the representation ρ_1 to ρ_4 ,

$$\phi(\{\pm 1\}) = \phi(\text{Ker}\rho_1) = \text{Ker}\rho_4 = \{1, \gamma\}.$$

In fact, triality induces a representation of the symmetric group Σ_3 on the center of $\text{Spin}(8)$, $\{\pm 1, \pm\gamma\}$, permuting the non-identity elements $\{-1, \gamma, -\gamma\}$, and ϕ acts as the transposition of the first two elements.

Thus

$$\begin{aligned} \text{Ker}(\rho_1\phi) \cap \text{Spin}(7) &= \phi^{-1}(\text{Ker}\rho_1) \cap \text{Spin}(7) \\ &= \phi(\text{Ker}\rho_1) \cap \text{Spin}(7) = \{1, \gamma\} \cap \text{Spin}(7) = \{1\}. \end{aligned}$$

This shows that $\rho_1\phi$ is injective on $\text{Spin}(7)$, and completes the proof that the stabilizer of ω_{Ca} in $\text{SO}(8)$ is isomorphic to $\text{Spin}(7)$. \square

The following classification by orbit type of comass 1 self-dual 4-forms (calibrating forms) is given in [DHM88].

- (1) Type $(1, 0)$, $\phi = \omega_{\text{Ca}}$, Cayley geometry;
- (2) Type $(2, 0)$, $\phi = \frac{1}{2}(\omega_{\text{Ca}} + \omega_2) = e^{1234} + e^{1256} + e^{1278}$, Kähler 4-form, that is, the square of the Kähler form;
- (3) Type $(3, 0)$, $\phi = \frac{1}{3}(\omega_{\text{Ca}} + \omega_2 + \omega_3) = \frac{1}{6}(\tau_I^2 + \tau_J^2 + \tau_K^2)$, Kraines form, quaternionic geometry;
- (4) Type $(1, 1)$, $\phi = \frac{1}{2}(\omega_{\text{Ca}} + \eta_4) = \text{Re}[(e_1 + ie_7)(e_2 - ie_8)(e_3 + ie_5)(e_4 - ie_6)]$, special Lagrangian geometry;
- (5) Type $(2, 1)$, $\phi = \frac{1}{4}\omega_{\text{Ca}} + \frac{1}{2}\omega_2 + \frac{1}{4}\eta_4$, $\mu = \frac{1}{4}\omega_{\text{Ca}} + \frac{1}{4}\omega_{\text{Ca}} + \frac{1}{2}\eta_4$, $\psi = \frac{1}{3}(\omega_{\text{Ca}} + \omega_2 + \eta_4)$, complex Lagrangian geometry;
- (6) Type $(2, 2)$, $\phi = \frac{1}{4}(\omega_{\text{Ca}} + \omega_2 + \eta_3 + \eta_4) = (e_{12} + e_{78})(e_{34} + e_{56})$;
- (7) Type $(3, 1)$, $\phi = \frac{1}{4}(\omega_{\text{Ca}} + \omega_2 + \omega_3 + \eta_4)$;
- (8) Type $(3, 2)$, $\psi = \frac{1}{5}(\omega_{\text{Ca}} + \omega_2 + \omega_3 + \eta_3 + \eta_4)$;
- (9) Type $(3, 3)$, $\mu = \frac{1}{6}(\omega_{\text{Ca}} + \omega_2 + \omega_3 + \eta_2 + \eta_3 + \eta_4)$.

7. A COUNTEREXAMPLE

One might have thought that for any choice of coefficients ± 1 in a linear combination of the forms

$$\{e^{1234}, e^{1256}, e^{1278}, e^{1357}, e^{1467}, e^{1368}, e^{1458}\}$$

would give a form of comass 1, which would, therefore, realize the maximal Wirtinger ratio 14. However, a calculation similar to that in the proof of Proposition 5.1 shows that the form ω_+ with all coefficients +1 has comass 2.

Proposition 7.1. *The self dual form*

$$\omega_+ = e^{1234} + e^{1256} + e^{1278} + e^{1357} + e^{1467} + e^{1368} + e^{1458}$$

has comass 2.

Proof. We have $\omega_+ = \frac{1}{2}\omega_2 - \frac{1}{2}\omega_4 + \frac{1}{2}\eta_1 + \frac{1}{2}\eta_3$. Thus the corresponding symmetric traceless matrix is

$$A_+ := \frac{1}{2}(A_2) - \frac{1}{2}(A_4) + \frac{1}{2}(-A_5) + \frac{1}{2}(-A_7),$$

where the A_j are defined above in the proof of Proposition 5.2; that is, $A_+ = D_+ + \frac{1}{8}I$ where

$$D_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As in the proof of Proposition 5.1

$$\begin{aligned} & \sup_{g \in \text{SO}(8)} (\omega_+, g(e_1 \wedge e_2 \wedge e_3 \wedge e_4)) = \\ &= \sup_{g \in \text{SO}(8)} (A_+, \phi(g)z_1) \\ &= \sup_{g \in \text{SO}(8)} \left(A_+ - \frac{1}{8}I, \phi(g)z_1 \right) \\ &= \sup_{g' = \phi(g) \in \text{SO}(8)} (D_+, g'z_1 g'^{-1}) \\ &\leq \frac{1}{2} \left(\sum_{j=2,4,5,7} \sup_{g' \in \text{SO}(8)} (D_j, g'z_1 g'^{-1}) \right) \\ &= 2, \end{aligned}$$

The last equality follows from the fact that the calculation of

$$\sup_{g' \in \text{SO}(8)} (D_1, g'z_1 g'^{-1}) = 1$$

in the proof of Proposition 5.1, applies equally well to the other D_j . The value 2 for $(D_+, g'z_1g'^{-1})$ is actually achieved for the matrix

$$g' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

A similar argument shows that the form

$$\begin{aligned} \mu &= e^{1234} - e^{1256} + e^{1278} + e^{1357} + e^{1467} + e^{1368} + e^{1458} \\ &= \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{1}{2}\eta_1 - \frac{1}{2}\eta_4 \end{aligned}$$

has comass 2.

Note that, by considering the associated diagonal matrices, and the action of the symmetric group, S_8 , it is clear that the comass 1 forms $\frac{1}{2}\omega_+ = \frac{1}{4}\omega_2 - \frac{1}{4}\omega_4 + \frac{1}{4}\eta_1 + \frac{1}{4}\eta_3$ and $\frac{1}{2}\mu = \frac{1}{4}\omega_2 + \frac{1}{4}\omega_3 + \frac{1}{4}\eta_1 - \frac{1}{4}\eta_4$ are $\text{SO}(8)$ -conjugate, respectively, to the convex combinations $\frac{1}{4}\omega_2 + \frac{1}{4}\eta_1 + \frac{1}{4}\eta_2 + \frac{1}{4}\eta_3$ and $\frac{1}{4}\omega_{\text{Ca}} + \frac{1}{4}\omega_2 + \frac{1}{4}\omega_3 + \frac{1}{4}\eta_1$, both of orbit type $(3, 1)$ in the classification of [DHM88].

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DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN 52900
 ISRAEL

E-mail address: {katzmik, shnider}@macs.biu.ac.il